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Regularity of Trace Rings of Generic Matrices

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INTRODUCTION

Throughout this paper, F is a field of characteristic zero, algebraically closed if necessary. With $\mathscr{P}_{m,n}$ we denote the polynomial algebra

$$F[x_{ij}(l) \mid 1 \leqslant i, j \leqslant n; 1 \leqslant l \leqslant m].$$

The F-subalgebra of $M_n(\mathscr{P}_{m,n})$ generated by the matrices

$$X_l = (x_{ij}(l))_{i,j}, \qquad \text{where} \quad 1 \leqslant l \leqslant m,$$

is called the ring of m generic $n \times n$ matrices $\mathbb{G}_{m,n}$. The F-subalgebra of $M_n(\mathscr{P}_{m,n})$ generated by $\mathbb{G}_{m,n}$ and $\mathrm{Tr}(\mathbb{G}_{m,n})$ is the ring of m generic $n \times n$ matrices and is denoted by $\mathbb{T}_{m,n}$.

These trace rings appear naturally in the study of finite dimensional representations of free algebras and in the invariant theory of $n \times n$ matrices, [7]. Unlike rings of generic matrices, $\mathbb{T}_{m,n}$ shares some properties with commutative polynomial rings; e.g., they are maximal orders and even unique factorization rings in the sense of Chatters and Jordan. However, their homological properties are far from being understood. The main aim of this paper is to prove the following result.

THEOREM. *If $n \leqslant 4$, then the trace ring of m generic $n \times n$ matrices has finite global dimension if and only if $n = 1$, $m = 1$, or $\mathbb{T}_{m,n} = \mathbb{T}_{2,2}$, $\mathbb{T}_{3,2}$, or $\mathbb{T}_{2,3}$.*

Pictorially, we have the situation

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n					
4	4	∞	∞	∞	
3	3	10	∞	∞	
2	2	5	9	∞	
1	1	2	3	4	
	1	2	3	4	m

Of course, the proof of the regularity of the commutative cases (i.e., $m=1$ or $n=1$) goes back to Hilbert. We were told that the first proof of the regularity of $\mathbb{T}_{2,2}$ was due to A. Schofield who showed that it can be written as a coproduct of two commutative polynomial rings, [5]. The first published proof is that of L. Small and T. Stafford [9]. They proved that $\mathbb{T}_{2,2}$ is an iterated Öre extension. In the first section we give an easy proof of this result based on the fact that $X_1X_2 - X_2X_1$ is a normalizing element in $\mathbb{T}_{2,2}$. Further, we show that also $\mathbb{T}_{3,2}$ has finite global dimension, using some results of C. Procesi [8], and that $\operatorname{gldim}(\mathbb{T}_{m,2}) = \infty$ for $m \geqslant 4$. The test we use throughout for regularity of positively graded Noetherian F-algebras whose part of degree zero is F is that its Poincaré series should be a pure inverse in $\mathbb{Z}[t]$. We give an example to show that this condition is not sufficient, in spite of an (erroneous) result of Govorov [6].

In the second section we prove the regularity of $\mathbb{T}_{2,3}$ and present an explicit description of it as a free module of rank 18 over a polynomial subring of its center. Moreover, we will show that $\operatorname{gldim}(\mathbb{T}_{m,3}) = \infty$ whenever $m \geqslant 3$.

In the final section, we prove that $\mathbb{T}_{m,4}$ can never be regular. The proof is based on a description of the Poincaré series of $\mathbb{T}_{m,n}$ due to E. Formanek [4, Th. 22] as a multivalued power series. We have not included the details of our computations since we believe there must exist an easier and more elegant way to compute the rational expression directly.

At this point we would like to make the conjecture that $\operatorname{gldim}(\mathbb{T}_{m,n}) = \infty$ whenever $m > 1$ and $n \geqslant 5$.

1. Generic 2×2 Matrices

First, we give an easy proof of the Formanek–Schofield or Small–Stafford result:

Proposition 1. $\operatorname{gldim}(\mathbb{T}_{2,2}) = 5$.

Proof. It is easy to verify that $\Delta = X_1X_2 - X_2X_1$ is a normalizing element of $\mathbb{T}_{2,2}$ and that the quotient is

$$\mathbb{T}_{2,2}/\Delta\mathbb{T}_{2,2} \simeq F[x_1, \operatorname{Tr}(x_1), x_2, \operatorname{Tr}(x_2)].$$

So, $\operatorname{gldim}(\mathbb{T}_{2,2}/\Delta\mathbb{T}_{2,2}) = 4$ and by a standard argument it follows that $\operatorname{gldim}(\mathbb{T}_{2,2}) = 5$.

In order to study the homological properties of $\mathbb{T}_{m,2}$ for $m \geqslant 3$ it is convenient to use the following result due to C. Procesi [8]:

$$\mathbb{T}_{m,2} = \mathbb{T}_m^0[\operatorname{Tr}(X_1), ..., \operatorname{Tr}(X_m)],$$

where $\mathbb{T}_m^0$ is the sub F-algebra of $\mathbb{T}_{m,2}$ generated by the generic trace zero matrices

$$X_i^0 = X_i - \tfrac{1}{2}\operatorname{Tr}(X_i),$$

where $1 \leqslant i \leqslant m$. It is well known that 2×2 trace zero matrices satisfy the commutation relation $AB + BA = \operatorname{Tr}(AB)$. So, if we define the generic Clifford algebra Cl_m to be the iterated Öre extension

$$F[a_{ij} \mid 1 \leqslant i < j \leqslant m][a_1][a_2, \sigma_2, \delta_2] \cdots [a_m, \sigma_m, \delta_m],$$

where $\sigma_j(a_i) = -a_i$ and $\delta_j(a_i) = a_{ij}$ for all $i < j$ and trivial actions on the other variables, then we get an epimorphism

$$\pi_m : \mathrm{Cl}_m \to \mathbb{T}_m^0$$

by sending a_i to X_i^0 and a_{ij} to $\operatorname{Tr}(X_i^0 X_j^0)$. Using this fact, it is now fairly easy to prove

PROPOSITION 2. $\operatorname{gldim}(\mathbb{T}_{3,2}) = 9$.

Proof. From its construction we obtain that $K\dim(\mathrm{Cl}_3) = \operatorname{gldim}(\mathrm{Cl}_3) = 6$. Since both Cl_3 and $\mathbb{T}_3^0$ are catenary algebras of the same Krull dimension, the epimorphism π_3 must be an isomorphism. Hence $\operatorname{gldim}(\mathbb{T}_{3,2}) = \operatorname{gldim}(\mathbb{T}_3^0[\operatorname{Tr}(X_1), \operatorname{Tr}(X_2), \operatorname{Tr}(X_3)]) = 9$.

A similar approach fails for $m \geqslant 4$. For example, if $m = 4$ then the Krull dimension of Cl_4 is 10, whereas that of $\mathbb{T}_4^0$ is 9. Therefore, $\operatorname{Ker}(\pi_4)$ must be a height one prime ideal of Cl_4 which are all generated by a normalizing element. We get

$$\operatorname{Ker}(\pi_4) = \mathrm{Cl}_4 \cdot S_4(X_1, X_2, X_3, X_4)$$

The Poincaré series of Cl_4 is readily seen to be $(1-t)^{-4}\cdot(1-t^2)^{-6}$ and since the kernel of π_4 is generated by a non-zerodivisor of degree 4 we get

$$\mathscr{P}(\mathbb{T}_4^0; t)=\frac{1-t^4}{(1-t)^4(1-t^2)^6}=\frac{1+t^2}{(1-t)^4(1-t^2)^5}.$$

$\mathbb{T}_{4,2}$ being a polynomial extension of $\mathbb{T}_4^0$ we find that its Poincaré series is not a pure inverse so its gloval dimension must be infinite. More generally, we have

PROPOSITION 3. *If* $m\geqslant 4$ *then* $\mathrm{gldim}(\mathbb{T}_{m,2})=\infty$.

Proof. Consider $\mathbb{T}_{m,2}$ in the natural way as an $\mathbb{N}^m$-graded F-algebra and suppose that it has finite global dimension, then its Poincaré series should be a pure inverse in $\mathbb{Z}(t_1, ..., t_m)$. The natural epimorphism $\mathbb{T}_{m,2}\to\mathbb{T}_{4,2}$ obtained by sending X_i to zero for $i\geqslant 5$ amounts on the level of the Poincaré series in a multigradation to

$$\mathscr{P}(\mathbb{T}_{4,2}; t_1, t_2, t_3, t_4)=\mathscr{P}(\mathbb{T}_{m,2}; t_1, ..., t_m)\mid t_5=0, ..., t_m=0$$

entailing that $\mathscr{P}(\mathbb{T}_{4,2}; t_1, t_2, t_3, t_4)$ should be a pure inverse, but we have seen above that this is impossible.

We think this is the proper place to show that for a positively graded affine F-algebra with part of degree zero, F-finite global dimension does not follows from the Poincaré series being a pure inverse. This is in spite of an (erroneous) result of Govorov [6].

EXAMPLE 4. Since $D(X^0)$ is a non-zerodivisor of degree two in $\mathbb{T}_2^0$ and $\mathscr{P}(\mathbb{T}_2^0; t)=(1-t)^{-2}(1-t^2)^{-1}$ we obtain

$$\mathscr{P}(\mathbb{T}_2^0/\mathbb{T}_2^0 D(X^0))=\frac{1-t^2}{(1-t)^2(1-t^2)}=\frac{1}{(1-t)^2}.$$

However, this ring cannot have finite global dimension for otherwise it had to be domain by a graded version of a result of Walker [11] and clearly $(X^0)^2=0$.

2. GENERIC 3 × 3 MATRICES

Before stating the main results of this section, we will recall some facts on skew-polynomial rings [2]. Suppose that R is a positively graded F-algebra with $R_0=F$ and R is generated by homogeneous elements $x_1, ..., x_r$ (not necessarely of degree one) satisfying the $r(r-1)/2$ relations

$$x_j\cdot x_i=\phi_{ij}\ (1\leqslant i<j\leqslant r),$$

where ϕ_{ij} is a sum of ascending monomials earlier than $x_j \cdot x_i$ in the lexicographic order which is induced by putting $x_i < x_j$ iff $i < j$.

If the overlap ambiguities $[x_i, x_j, x_k]$ for $k > j > i$ are constant then one can apply Bergman's diamond lemma [3] in order to get that R has an F-basis consisting of monomials of the form

$$x_1^{a_1} \cdots x_r^{a_r}$$

for natural numbers $a_1, ..., a_r$. Annick's resolution of F (see [1]) then shows that F has finite projective dimension. Therefore, if R is Noetherian, then R has finite global dimension.

LEMMA 5 (Nakayma's lemma for graded rings [12]). *Let R be a positively graded F-algebra with $R_0 = F$ and let M be a graded R-module with left bounded grading but not necessarily finitely generated, then if $R^+ M = M$ then $M = 0$.*

The proof is obvious. An an immediate consequence of this we get

COROLLARY 6. *With the same assumptions as above suppose that $M/R^+ M$ is generated by the images of $m_1, ..., m_t \in M$, then there elements generate M.*

Proof. Let N be the cokernel of the natural map $R^t \to M$ obtained by sending the ith basisvector to m_i. Clearly, N has a left bounded grading and $N = R^+ N$ by the assumptions, so $N = 0$.

We are now in a position to prove the main theorem of this section:

PROPOSITION 7. *The trace ring of two generic* 3×3 *matrices has global dimension* 10.

Proof. Since $\mathbb{T}_{2,3} = \mathbb{T}^0[\mathrm{Tr}(X_1), \mathrm{Tr}(X_2)]$, where $\mathbb{T}^0$ is the trace ring of two generic trace zero 3×3 matrices, it is enough to prove that $\mathrm{gldim}(\mathbb{T}^0) = 8$. If X and Y denote two generic trace zero 3×3 matrices, then the homogeneous pieces of the (multigraded) Cayley–Hamilton polynomial of $X + Y$ give us the following relations:

$$g_1: \; X^3 + CX + F = 0$$

$$g_2: \; X^2Y + XYX + YX^2 + CY + DX + H = 0$$

$$g_3: \; Y^2X + YXY + XY^2 + DY + EX + G = 0$$

$$g_4: \; Y^3 + EY + I = 0,$$

where

$$C = -\tfrac{1}{2}T(X^2); \qquad D = -T(XY); \qquad E = -\tfrac{1}{2}T(Y^2)$$
$$G = -T(XY^2); \qquad H = -T(YX^2)$$
$$F = -\tfrac{1}{3}T(X^3); \qquad I = -\tfrac{1}{3}T(Y^3).$$

We define Λ to be the F-algebra

$$F[C, D, E, F, G, H, I]\langle X, Y\rangle/(g_1, g_2, g_3, g_4).$$

Since g_1 and g_4 only express that $X^3 + CX$ and $Y^3 + EY$ are central, and since this can also be deduced from g_2 and g_3 we know that Λ is also the F-algebra

$$F[C, D, E, G, H]\langle X, Y\rangle/(g_2, g_3).$$

If we choose the lexicographic ordering $Y > X$ then it is easy to check that the overlap between the leading terms of g_2 and g_3 gives no extra relations. Therefore, Λ has a basis of reduced monomials

$$C^a D^b E^c G^d H^e X^f (YX)^g Y^h,$$

whence Λ is a skew-polynomial ring. In order to prove that $\mathrm{gldim}(\Lambda) = 8$ it suffices to prove that Λ is a finite module over a Noetherian commutative subring. With J we denote the element

$$2XYXY + X^2Y^2 + YX^2Y + YXYX + XY^2X$$
$$+ 2DXY + DYX + GX + HY.$$

A straightforward but tedious calculation shows that J is a central element in Λ. Since the overlaps $[Y, YX, YX]$, $[YX, YX, X]$, and $[YX, YX, YX]$ give no extra replacements we get than an F-vectorspace basis of

$$\Lambda/(C, D, E, F, G, H, I, J)$$

is given by

$$X^{\varepsilon_1}(YX)^{\varepsilon_2} Y^{\varepsilon_3},$$

where $\varepsilon_1, \varepsilon_3 \in \{0, 1, 2\}$ and $\varepsilon_2 \in \{0, 1\}$. So by Corollary 6 we get that Λ is a finite module over the polynomial ring

$$R = F[C, D, E, F, G, H, I, J].$$

Hence we have proved that Λ is a graded local order of finite global dimension. From now on we will liberally use graded versions of theorems in [10], which are routine exercises.

In particular we deduce that the center of Λ is integrally closed and hence that Λ is closed under taking traces. Since Λ is regular it is Cohen–Macaulay by [10] and hence $rk_R(\Lambda) = 18$ the p.i.-degree of Λ must be equal to 3. There is a natural map

$$\phi : \Lambda \to \mathbb{T}^0$$

by sending X to $X_1 - \frac{1}{3}\operatorname{Tr}(X_1)$ and Y to $X_2 - \frac{1}{3}\operatorname{Tr}(X_2)$. This maps splits since Λ is of p.i.-degree 3 and closed under taking traces. So, ϕ is surjective, whence an isomorphism since both affine F-algebras have the same Krull dimension. So $\mathbb{T}^0 = \Lambda$ which finishes the proof.

PROPOSITION 8. *The trace ring of two generic* 3×3 *matrices is a free module of rank* 18 *over a polynomial subring of the center.*

Proof. Since $\operatorname{gldim}(\mathbb{T}^0) = 8$ we know that $\mathbb{T}^0$ is Cohen–Macaulay [10], so $\operatorname{depth}(\mathbb{T}^0) = 8$ and hence

$$\{C, D, E, F, G, H, I, J\}$$

is a regular sequence, finishing the proof.

Another regular sequence for Λ can be obtained as follows. Let

$$\Gamma = \Lambda/(C, D, E, G, H),$$

then $Z = YX - \omega XY$, where ω is a primitive 3rd root of unity is a normalizing element in Γ. Dividing out Z we end up with the cyclic algebra

$$\begin{pmatrix} I & & F \\ & F(I, F] & \end{pmatrix}$$

that is $X^3 = F$, $Y^3 = I$, and $YX = \omega XY$. Hence X and Y complete the regular sequence

$$\{C, D, G, E, H, Z\};$$

i.e., $\mathbb{T}^0$ and also $\mathbb{T}_{2,3}$ are even regular in the sense of Walker.

The above discussion also enables us to compute the rational expression of the Poincaré series of $\mathbb{T}_{2,3}$:

$$\mathscr{P}(\mathbb{T}_{2,3}; s, t) = \frac{1}{(1-s)^2(1-t)^2(1-s^2)(1-t^2)(1-st)^2(1-s^2t)(1-st^2)}.$$

Procesi has proved in [7] that the center of the trace ring of m generic $n \times n$ matrices $\mathscr{R}_{m,n}$ is affine and is generated as an F-algebra by the elements

$$\operatorname{Tr}(X_{i_1} \cdots X_{i_j}); \qquad j \leqslant 2^n - 1$$

and the indices i_k range from 1 to m. Since $\mathbb{T}_{m,n}$ is a finite module over $\mathscr{R}_{m,n}$ this entails that there is a symmetric polynomial $f(t_1, \dots, t_m)$ such that

$$\mathscr{P}(\mathbb{T}_{m,n}; t_1, \dots, t_m) = \frac{f(t_1, \dots, t_m)}{\prod (1 - t_i) \prod (1 - t_i t_j) \cdots \prod (1 - t_{i_1} \cdots t_{i2^n - 1})}.$$

Using this fact is now fairly easy to prove

PROPOSITION 9. *For* $m \geqslant 3$, $\operatorname{gldim}(\mathbb{T}_{m,3}) = \infty$.

Proof. As in the proof of Proposition 3 it suffices to show that $\mathscr{P}(\mathbb{T}_{3,3}; t_1, t_2, t_3)$ cannot be a pure inverse. So, let us assume that it is a pure inverse, then it has the form

$$\frac{1}{\prod g_i(t_1, t_2, t_3)},$$

where each of the $g_i(t_1, t_2, t_3)$ is an irreducible factor in $\mathbb{Z}[t_1, t_2, t_3]$ of $1 - t_1^k t_2^l t_3^m$ with $k + l + m \leqslant 2^n - 1$. Let us look at the subproduct of the factors containing only two indeterminates t_i and t_j, then after specializing the remaining indeterminate to zero this subproduct must be equal to

$$(1 - t_i)^2 (1 - t_j)^2 (1 - t_i^2)(1 - t_j^2)(1 - t_i t_j)^2 (1 - t_i^2)(1 - t_i t_j^2)$$

since its inverse must be equal to $\mathscr{P}(\mathbb{T}_{2,3}; t_i, t_j)$. Therefore, since this holds for any couple $i \neq j$ from $\{1, 2, 3\}$ we obtain a subproduct factor

$$(1 - t_1)^2 (1 - t_2)^2 (1 - t_3)^2 (1 - t_1^2)(1 - t_2^2)(1 - t_3^2)$$
$$(1 - t_1 t_2)^2 (1 - t_1 t_3)^2 (1 - t_2 t_3)^2$$
$$(1 - t_1^2 t_2)(1 - t_1^2 t_3)(1 - t_2^2 t_3)(1 - t_2^2 t_1)(1 - t_3^2 t_1)(1 - t_3^2 t_2).$$

Now, change again to a single gradation, then the order of the pole of $\mathscr{P}(\mathbb{T}_{2,3}; t)$ at $t = 1$ is at least 21 by the above argument. However, we know that this order must be equal to the Krull dimension of $\mathbb{T}_{2,3}$ which is 19, a contradiction finishing the proof.

3. Generic 4×4 Matrices

Using the description of Formanek [4] of the Poincaré series of trace rings of generic matrices, it is an easy but boring job to calculate the first terms in the power series expansion of $\mathscr{P}(\mathbb{T}_{2,4}; s, t)$. We obtain

$$\begin{gathered}1 + \\ 2s + 2t + \\ 4s^2 + 6st + 4t^2 + \\ 7s^3 + 14s^2t + 14st^2 + 7s^3 + \\ 11s^4 + 27s^3t + 37s^2t^2 + 27st^3 + 11t^4 + \\ 16s^5 + 46s^4t + 77s^3t^2 + 77s^2t^3 + 46st^4 + 16t^5 + \\ 23s^6 + 72s^5t + 141s^4t^2 + 174s^3t^3 + 141s^2t^4 + 72st^5 + 23s^6 + \\ 31s^7 + 107s^6t + 233s^5t^2 + 338s^4t^3 + 338s^3t^4 + 233s^2t^5 + 107st^6 + 31t^7 + \end{gathered}$$

Or, in a single gradation we obtain that

$$\mathscr{P}(\mathbb{T}_{2,4}; t) = 1 + 4t + 14t^2 + 42t^3 + 113t^4 + 278t^5 + 646t^6 + 1418t^7 + \cdots.$$

Using this information we can prove the next

Proposition 10. *For all $m \geqslant 2$, $\operatorname{gldim}(\mathbb{T}_{m,4}) = \infty$.*

Proof. Of course, it is sufficient to show that the Poincaré series of the trace ring of two generic 4×4 matrices is not a pure inverse. Now, suppose it is a pure inverse, then its denominator consists of a product of irreducible factors in $\mathbb{Z}[s, t]$ of elements of the form $1 - s^i t^j$, where $i + j \leqslant 15$. Since

$$\mathscr{P}(\mathbb{T}_{1,4}; x) = \frac{1}{(1-x)^2(1-x^2)(1-x^3)}$$

we know that the subfactor $F(s, t)$ of the products of factors of elements containing just one of the indeterminates is equal to

$$F(s, t) = (1-s)^2(1-t)^2(1-s^2)(1-t^2)(1-s^3)(1-t^3).$$

So, we divide the Poincaré series by $F(s, t)^{-1}$. The resulting power series (in single gradation) is of the form

$$1 + 2t^2 + 4t^3 + 7t^4 + 10t^5 + 23t^6 + 38t^7 + \cdots.$$

Next, let us consider the subfactor $G(st)$ which is the product of all factors of elements of the form $1-(st)^k$, where k is necessarily $\leqslant 7$. Then $G(st)$ can be brought into the form

$$(1+st)^{-a}(1-st)^{\alpha}(1-s^2t^2)^{\beta}(1-s^3t^3)^{\gamma}(1-s^4t^4)^{b}$$
$$\times(1-s^5t^5)^{c}(1-s^6t^6)^{d}(1-s^7t^7)^{e},$$

where $a, b, c, d, e \in \mathbb{N}$, whereas $\alpha, \beta, \gamma \in \mathbb{Z}$.

One can continue in this way; e.g., the subfactor $H(s^2t, st^2)$ consisting of all products of factors of elements of the form $1-(s^2t)^k$ or $1-(st^2)^k$, where k is necessarily $\leqslant 5$ can be brought into the form

$$[(1-s^2t)(1-t^2s)]^{\alpha}[(1-s^4t^2)(1-t^4s^2)]^{\beta}[(1-s^6t^3)(1-t^6s^3)]^{a}$$
$$[(1-s^8t^4)(1-t^8s^4)]^{b}[(1-s^{10}t^5)(1-t^{10}s^5)]^{c},$$

where $a, b, c \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{Z}$.

Ultimately, one can show in this way that the rational expression of $\mathscr{P}(\mathbb{T}_{2,4}; s, t)\cdot F(s, t)$ in a single gradation (i.e., putting $s=t$) can be written as

$$(1+t^2)^{a}(1-t^2)^{-\alpha}(1-t^3)^{-\beta}(1-t^4)^{-\gamma}(1-t^5)^{-\delta}$$
$$(1+t^6)^{-\varepsilon}(1-t^7)^{-\omega}(1-t^8)^{-b}(1-t^9)^{-c}(1-t^{10})^{-d}$$
$$(1-t^{11})^{-e}(1-t^{12})^{-f}(1-t^{13})^{-g}(1-t^{14})^{-h}(1-t^{15})^{-i},$$

where Latin letters are in $\mathbb{N}$ and Greek ones in $\mathbb{Z}$.

Let us first assume that $\alpha \geqslant 0$, then comparing the power series expansion of this expression with the one obtained above we get that $a+\alpha=2$. Therefore, we have to investigate three cases: (1) $\alpha=2$ and $\alpha=0$ then we get $\beta=4$, $\gamma=6$, $\delta=2$, $\varepsilon=1$, and $\omega=6$. Therefore, the pole of $\mathscr{P}(\mathbb{T}_{2,4}; t)\cdot F(t)$ is at least 19 in $t=1$, whence of $\mathscr{P}(\mathbb{T}_{2,4}; t)$ at least 27. However, this order must be equal to the Krull dimension of $\mathbb{T}_{2,4}$ which is 17, a contradiction.

(2) $a=1$ and $\alpha=1$, then we get $\beta=4$, $\gamma=5$, $\delta=2$, $\varepsilon=1$, and $\omega=6$ yielding that the order of the pole of $\mathscr{P}(\mathbb{T}_{2,4}; t)$ in $t=1$ is at least 27, a contradiction.

(3) $a=0$ and $\alpha=2$, then we get that $\beta=4$, $\gamma=4$, $\delta=2$, $\varepsilon=1$ and $\omega=6$ and again the order is at least 27, a contradiction.

Of course, the remaining possibility is that $\alpha<0$ reduces to case (1), finishing the proof.

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